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Evidence for S-Duality in N=4 Supersymmetric Gauge Theory

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Abstract

Using D-brane techniques, we compute the spectrum of stable BPS states in N=4 supersymmetric gauge theory, and find it is consistent with Montonen-Olive duality.

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Gauge theory in four dimensions with $N = 4$ supersymmetry has been conjectured by Montonen and Olive to possess a strong/weak coupling duality symmetry [1,2]. For the simplest case, with $SU(2)$ gauge symmetry, the duality group is $SL(2, \mathbb{Z})$. By considering the orbits of the states in the $N = 4$ gauge multiplet under this $SL(2, \mathbb{Z})$, a whole tower of BPS[#] monopole and dyon states are predicted with magnetic-electric charges given by (p, q) with p and q relatively prime [3]. Proving the existence of these states is an important check of the Montonen-Olive S-duality conjecture.

The states with magnetic charge one are the usual 't Hooft-Polyakov monopole [4] and the Julia-Zee dyons [5] embedded in $N = 4$ gauge theory. Sen showed the states with magnetic charge two (and odd electric charge) arise as bound states of two singly charged monopoles [3]. States with magnetic charge larger than two will arise as bound states of multi-monopole configurations. The construction of these states as bound states in the supersymmetric quantum mechanics on the moduli space of these configurations remains an open problem (see [6] for results in this direction). In this paper, we will take a different approach and use the recently developed D-brane technology [7] to prove the existence of the whole tower of (p, q)-states with degeneracies in accord with S-duality. We also show that this generalizes in a straightforward way to ADE groups.

The starting point for this construction is the Type IIB string theory compactified on $K3 \times T^2$ [8–11]. This gives rise to a theory in four dimensions with $N = 4$ supersymmetry. One considers a point in the $K3$ moduli space near which the Type IIA string develops an enhanced gauge symmetry [12,13]. In this limit the $K3$ develops an orbifold singularity of ADE type as a set of two-spheres shrink to zero size. Type II string theory compactified on this and closely related manifolds have recently been considered in [14–17], and D-branes on such manifolds have been studied in [10,15,18,19]. From the Type IIB perspective the relevant BPS states arise from self-dual supersymmetric threebranes [20] wrapping the supersymmetric three-cycles [21] of $K3 \times T^2$ which are shrinking to zero size. These three-cycles are the product of the two-sphere (which is a holomorphic two-cycle of the $K3$ with respect to an appropriate choice of complex structure) with a one-cycle of the torus. In string units the mass of such a state is

$$M \sim \frac{\epsilon R}{\lambda} \ , \tag{1}$$

where ϵ is the area of the S^2, R is the size of the one-cycle of the torus and λ is the Type IIB string coupling constant. Since ϵ can be made arbitrarily small by moving toward the orbifold singularity of the $K3$, we may consider a limit in which gravity, the other fundamental IIB string excitations, and the Kaluza-Klein modes, are irrelevant. The effective four-dimensional theory describing these threebrane states will be some $N = 4$ gauge theory [9], valid up to some cutoff $\Lambda = \min(1/R, \sqrt{\epsilon/\lambda})$.*

[#] In this paper we will reserve the term BPS for states which preserve one half of the supersymmetry.

* This cutoff could be removed by going to the point-particle limit along the lines of [14](also see final paragraph).

Let us first consider the case when a single two-sphere shrinks to zero size, corresponding to $SU(2)$ gauge theory. The $U(1)$ gauge field A of this theory arises from the 4-form potential C of the IIB theory

$$C = A \wedge G \wedge dx \ , \tag{2}$$

where G is a self-dual harmonic two-form on $K3$ with support on the neighborhood of the S^2, and $z = x + iy$ is the complex coordinate of T^2, with periodic identification $z \sim z + 1$, $z \sim z + \tau$. Note that because C is the potential for a self-dual field strength, only one independent $U(1)$ gauge field appears in four dimensions.

We may regard the states wrapping an A-cycle (i.e. in the x-direction) of T^2 q-times and the B-cycle p-times as carrying magnetic and electric charge (p, q). The BPS condition requires the three-cycle have minimal volume, which in turn implies that these (p, q)-cycles are straight lines in the z-plane. The masses of these states are therefore

$$M = \frac{\epsilon R}{\lambda} |p + \tau q| \ . \tag{3}$$

We see the coupling constant of the gauge theory is to be identified with the complex structure parameter τ of the torus. Note the $SL(2, \mathbb{Z})$ T-duality symmetry of the Type IIB which acts on τ is mapped to the S-duality symmetry of the four-dimensional gauge theory. Proving T-duality of the nonperturbatively defined IIB string theory would then amount to proving S-duality of $N = 4$ gauge theory.

To complete this check of the S-duality conjecture we must compute the degeneracy of these (p, q)-states. To accomplish this we will move to a point in the moduli space of the IIB theory at which the three-cycle becomes very large, so that the mass of the solitonic states becomes enormous compared to the masses of the fundamental string excitations. Since we have $N = 4$ supersymmetry, we know the degeneracy of these states will not change as we move in the moduli space. Type IIB string perturbation theory will be valid in this limit, and we may use the D-brane picture [7] to describe these threebrane states.

In this limit we can use the $R \to 1/R$ duality of perturbative string theory acting on the circle of the two-torus which is defined by (p, q). This will also map type IIB to type IIA theory and three-branes to two-branes. If the greatest common divisor of p and q is n, then the corresponding state will be mapped to n two-branes of the type IIA theory wrapping the S^2 of the $K3$. In this situation the number of BPS states which preserve one half the supersymmetry, provided by D-brane configurations, is a topological invariant (related to the dimension of the cohomology of the moduli space of the corresponding Hitchin system [22]). For $n > 1$ we may regard the n D-branes wrapping the S^2 as n copies of the S^2, i.e. as a two-cycle with self-intersection number $-2n^2$ [22]. It may be shown that such a cycle can not be realized as a smooth holomorphic curve, hence no supersymmetric bound states will be produced. For $n = 1$, the S^2 can always be realized as a smooth genus zero holomorphic curve and the moduli space of the relevant Hitchin system is a point. The dimension of the cohomology is one, so a single supersymmetric bound state exists. This completes the argument.

These results for $SU(2)$ extend in a straightforward way to higher rank gauge theories

of ADE type.[♭] S-duality predicts a single BPS multiplet with magnetic-electric charges $(p,q)\boldsymbol{\alpha}$ for each root $\boldsymbol{\alpha}$ of the group, with p,q relatively prime. In the D-brane picture, these will arise from threebranes partially wrapping two-spheres shrinking to zero size as an orbifold singularity (of A, D or E type) of the $K3$ is approached.

For our purposes it is sufficient to study the behavior of the geometry near the singularity, which allows us to replace the $K3$ by an ALE space $\mathcal{M}$ of type G [23]. The homology $H_2(\mathcal{M},\mathbb{Z}) \cong \mathbb{Z}^r$ (r is the rank of G) and the intersection form is $-C_G$, the Cartan matrix for the group G, thus the integral homology is identified with the root lattice of the group. The cohomology group $H^2(\mathcal{M},\mathbb{Z})$ is identified with the weight lattice of G and corresponds to a set of anti-self-dual two-forms. In addition, there are three covariantly constant self-dual two-forms $\vec{\omega}$ corresponding to the Kähler form and the holomorphic two-form (with respect to some choice of complex structure). These self-dual forms are non-normalizable on the ALE space. We denote the periods of these three two-forms by

$$\int_{\Sigma_i} \vec{\omega} = \vec{\zeta}_i \ , \tag{4}$$

for some choice of basis Σ_i ($i = 1,\cdots,r$) of $H_2(\mathcal{M},\mathbb{Z})$, corresponding to a choice of simple roots. The area of a two-cycle corresponding to a root $\boldsymbol{\alpha}$ will be

$$\epsilon \propto |\boldsymbol{\alpha}\cdot\vec{\zeta}| \ . \tag{5}$$

When we consider the dyon states associated with a simple root, the D-branes will wrap a single two-sphere Σ_i and the above results for the $SU(2)$ case will go through. For states associated with nonsimple root $\boldsymbol{\alpha}$, the D-branes will wrap the product of a collection of intersecting two-spheres with a single S^1 of the two-torus. For generic values of the moduli of the ALE space we may nevertheless regard this collection of intersecting two-spheres as a single two-sphere. This may be realized as a smooth holomorphic genus zero curve, and we may therefore again apply the preceding arguments which show that one BPS multiplet with charge $(p,q)\boldsymbol{\alpha}$ exists for every p,q relatively prime. Recalling that the moduli of the ALE space are simply the $\vec{\zeta}_i$ modulo the Euclidean group in three dimensions, one sees from (5) that for generic values of the moduli these states will be true bound states. However a subtlety appears when we adjust the moduli so that two or more of the $\vec{\zeta}_i$ are parallel. At this point a nonsimple root state may be only neutrally stable with respect to decay into simple root states. In this limit, the corresponding two-cycle in the D-brane picture will become degenerate, corresponding to two two-spheres intersecting at a point, for example. Using the fact that no jumping phenomena can occur in theories with $N = 4$ supersymmetry, we may nevertheless argue that a bound state at threshold must appear.

This agrees with recent field theory results which construct these states at threshold as bound states in the supersymmetric quantum mechanics on the moduli space of monopoles of distinct type [24]. The moduli spaces in question were constructed for a single real Higgs

[♭] Note that since the group must appear as an orbifold singularity of $K3$ we are restricted to groups with rank ≤ 19 (or 20 if we include nontrivial B-fields).

field. We expect that for generic asymptotic vevs of two or more Higgs fields,$^\natural$ the analogous moduli spaces will collapse to a simple $R^3 \times S^1$ form.

We must also show no other bound states arise in the D-brane picture. Clearly no new bound states will come from D-threebranes wrapping disconnected sets of two-spheres, so we need only consider a connected set corresponding to some point on the root lattice. In order to get a supersymmetric configuration, it is necessary to take the product of the two-spheres with parallel S^1's on the torus, as opposed to different nonparallel S^1's for different two-spheres. One may then move to a point in moduli space where IIB perturbation theory applies and apply $R \to 1/R$ duality in the S^1 direction defined by (p, q) (with p,q relatively prime). This yields a configuration equivalent to a single D-twobrane wrapping some element $\Sigma \in H_2(\mathcal{M}, \mathbb{Z})$. This element will have self-intersection number $\Sigma^2 \leq -2$. However such an element may only be realized as a holomorphic two-cycle when $\Sigma^2 \geq -2$, thus only the roots of the gauge group (which are in one-to-one correspondence with the $\Sigma^2 = -2$ elements) give rise to supersymmetric bound states, as described above.

Finally, let us notice that by going to the point particle limit of the string [14], it is possible to obtain a rigid field theory valid for all energy scales. The point particle limit fixes the $K3$ moduli parameters at the values where it develops the ADE orbifold point, and resolves this singularity by blowing up the singular $K3$ to an ALE space $\mathcal{M}$. We can fix the Kähler form of the ALE space in such a way that the Higgs fields ϕ take only values in a two-dimensional subspace. In this situation the $\vec{\zeta}_i$ are restricted to move in an R^2 subspace and their configuration is isomorphic to that defined by the set of points over the complex plane

$$V = \{x | P_G = \det(\phi - x) = 0\}. \tag{6}$$

Therefore the information contained in $\mathcal{M} \times T^2$ is alternatively encoded in the curve defined by the trivial fibration of the set V over the torus T^2. This curve coincides with the one proposed in [25,26] for representing the moduli space of $N = 4$ Yang-Mills theory with gauge group G. In the same way as for $N = 2$ string theories [15] we see that for $N = 4$ it is also possible to recover from the string compactification, in addition to the effective gauge theory physics, its associated Seiberg-Witten curve.

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$^\natural$ With $N = 4$ supersymmetry we have a total of six adjoint Higgs fields present. These give rise to moduli corresponding to the moduli of the ALE space, with B-field, together with additional moduli coming from RR fields. Note that only a subset of these moduli (related to the $\vec{\zeta}_i$) show up in our present construction, based on classical geometry.

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